Capillary pressure in a porous medium with distinct pore surface and pore volume fractal dimensions

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The relationship between capillary pressure and saturation in a porous medium often exhibits a power-law dependence. The physical basis for this relation has been substantiated by assuming that capillary pressure is directly related to the pore radius. When the pore space of a medium exhibits fractal structure this approach results in a power-law relation with an exponent of $3-D_v$, where D_v is the pore volume fractal dimension. However, larger values of the exponent than are realistically allowed by this result have long been known to occur. Using a thermodynamic formulation for equilibrium capillary pressure we show that the standard result is a special case of the more general exponent $(3-D_v)/(3-D_s)$ where D_s is the surface fractal dimension of the pores. The analysis reduces to the standard result when $D_s=2$, indicating a Euclidean relationship between a pore's surface area and the volume it encloses, and allows for a larger value for the exponent than the standard result when $D_s>2$.

I. INTRODUCTION

Accurately predicting capillary pressures in a porous medium is central to understanding the movement and distribution of fluids within such systems. The relationship between these pressures and fluid saturation often follows some form of power law where $\theta/\theta_s \simeq (P_s/P)^\lambda$ (e.g., [1,2]). Here θ is the moisture content, P is the capillary pressure, s denotes saturation, and λ is an empirically determined exponent. This relation was first established empirically, but subsequent analysis provided it with a physical basis by assuming that both capillary pressure and pore volume are directly proportional to the pore radius. If a system's pore space has fractal structure, this approach then yields a power-law relation between capillary pressure and saturation where the exponent λ is equal to $3-D_v$, with D_v being the pore volume fractal dimension (e.g., [3–8]). However, larger values of the exponent than are realistically allowed by this result have long been known to occur (e.g., [1]).

Recent work on microfluidic cells suggests that it is not so much pore diameter that determines capillary pressure, but fluid interfacial area [9]. Subsequent analysis has shown that the variation in fluid interfacial areas with respect to fluid volume is a significant factor determining capillary pressure under equilibrium conditions [10]. The physical structure of a pore network is particularly relevant in this context because it will directly affect the interfacial areas.

It is well established that the pore space of a wide variety of media exhibits fractal structure (e.g., [11]). It is also known that the surfaces of the grains and particles that com-prise porous systems can themselves be fractal [12,13]. A few studies have recently measured both D_v and the surface fractal dimension D_s for the same porous medium in an attempt to see if and how they differ. Thin slices from eight different kinds of unconsolidated media, both artificial and natural, were analyzed by Gimenez *et al.* [14]. Using digital images with a resolution of 60 μm they found D_v and D_s to differ by 0.25–0.56 with the pore volume fractal dimension exceeding the surface one. Using pulsed gradient NMR, Stallmach *et al.* [15] found the pore surface and pore volume fractal dimensions of a glacial sand to differ by 0.7 ± 0.05. A recent study by Dathe and Thullner [16] analyzed thin slices of two types of undisturbed soil structure using a scanning electron microscope with a resolution of 0.5 μm and produced similar results. Other authors have also obtained data showing that distinct fractal dimensions apply to different parts of a porous system (i.e., solid matrix, pore space or grain surface) (e.g., [17–19]).

In the present work we extend our thermodynamic formulation of equilibrium capillary pressure [10] to a system where the pore volume and pore surface fractal dimensions are taken into account. The analysis leads to a power-law relationship between equilibrium capillary pressure and moisture content where the power-law exponent is given by $(3-D_v)/(3-D_s)$. This result allows for the full range of observed exponents and reduces to the standard result when $D_s=2$ and gives a larger value for the exponent than the standard result when $D_s>2$. The analysis also indicates that a power-law relation between moisture content and capillary pressure will only hold for an equilibrium situation.
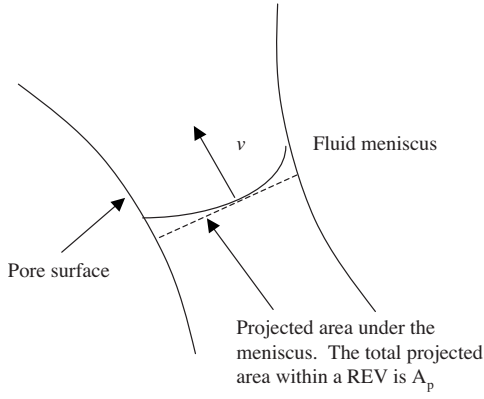
FIG. 1. *Fluid meniscus in motion.* The last term on the right-hand side of Eq. (1) applies to situations where the fluid-solid contact lines with a representative elementary volume (REV) are in motion. Several studies suggest that such movement results in non-equilibrium capillary pressures (e.g., [10,20,21]).

II. POWER-LAW RELATION BETWEEN CAPILLARY PRESSURE AND SATURATION IN A POROUS MEDIUM

Using a control mass formulation, the authors previously developed [10] a relation for the internal energy of an element of wetting fluid that enters a region of porous media under pressure p_w and distributes itself through a quasistatic process by displacing a nonwetting fluid that is at pressure p_n:

$$U' = TS' - p_w V_w' - p_n V_n' + \sum \sigma_{w,i} A_{w,i}' + \langle \alpha \mathbf{F} \cdot \boldsymbol{v} A_p v \rangle. \quad (1)$$

Equation (1) adopts the convention that fluid pressure, and related parameters, are quantities averaged over a representative elementary volume (REV) within the soil matrix. Here U is the internal energy, T is the temperature, S is the entropy of the wetting fluid, p_w is the pressure in the wetting phase, p_n is the pressure in the nonwetting phase, V_w is the volume of the wetting fluid, V_n is the volume of the nonwetting fluid, and $A_{w,i}$ is the interfacial area of the wetting fluid in contact with the ith nonwetting phase, $\sigma_{w,i}$ is the respective surface tension, and the primes denote the derivative with respect to time. The last term on the right applies to situations where the fluid-solid contact lines with a representative elementary volume (REV) are in motion. Several studies suggest that such movement would result in nonequilibrium capillary pressures (e.g., [10,20,21]). Here α is the total fluid-solid contact line length, F is the force on that line per unit of its length and velocity, $\boldsymbol{v}$ is the velocity of that contact line, v is its magnitude, and A_p is the projected interfacial area of the fluid, and the brackets denote an average over a REV, Fig. 1.

Under equilibrium conditions the authors showed [10] that capillary pressure can be written as

$$P_c = - \sum \sigma_{w,i} (\partial A_{w,i} / \partial V_w)_{U,S,T}. \quad (2)$$

At any given saturation in a porous medium under equilibrium conditions, small pores fill first with larger ones doing so when their capillarity can overcome the tension in the fluid surrounding them. A consequence of this in a system with a fractal pore volume distribution is that changes in

fluid volume correspond to the filling or emptying of pores at only one scale at a time. In order to evaluate the derivatives in Eq. (1) a relation is therefore needed to give a fluid's interfacial area in terms of its volume within those pores. For a system where the pore surfaces exhibit fractal structure we follow Gimenez *et al.* [14] and use Mandelbrot's relation between a self similar fractal area and the volume that it encloses ([22], p. 112):

$$A \propto V^{D_s/3}. \quad (3)$$

Here A is the surface area of a given pore, V is the volume it encloses, and $2 < D_s < 3$. Equation (3) is the "area-volume" extension of the slit-island relation ([21], p. 112 and [22]) which has previously been shown to hold in a porous medium with fractal pore boundaries [14]. We make the relation in Eq. (3) direct by introducing a constant c such that $A = cV^{D_s/3}$. Assuming that fluid volume is equal to pore volume to a negligible error and that the interfacial areas $A_{w,i}$ follow the proportionality in Eq. (3), $A_{w,i} = f_{w,i} V^{D_s/3}$ where $f_{w,i}$ is a shape factor that remains constant across all scales. Using this relation we can write $(\partial A_{w,i} / \partial V_w) = f_{w,i} (D_s/3) V^{(1/3)(D_s-3)}$, where V is the volume of the largest pore filled. For a situation where a wetting fluid displaces air in a porous medium, Eqs. (1)–(3) can be combined to give

$$P_c = - (f_{\text{fluid,air}} \sigma_{\text{fluid,air}} + f_{\text{fluid,solid}} \sigma_{\text{fluid,solid}})(D_s/3) V^{(1/3)(D_s-3)}. \quad (4)$$

When $D_s = 2$, Eq. (4) reduces to give the $1/r$ dependence that is typically assumed for capillary pressure [6–8,24].

III. MOISTURE CONTENT AND CAPILLARY PRESSURE

The moisture content within a two-phase system at equilibrium can be written as $V_w = \int V dN$, where V_w is the fluid volume within a REV, V is the volume of a single filled pore of a given size, N is the number of those pores within the REV, and the integral is over the applicable size range. If the pore volume is distributed in a fractal manner, we can follow Perrier *et al.* [8] who used the cumulative distribution function $Nr(l > L) = FL^{-D_v}$ to determine total pore volume, where L is taken to be the radius that corresponds to V, F is a constant, and D_v is the pore volume fractal dimension. Noting that $L \propto V^{1/3}$ it can then be shown that

$$V_s - V_w = (FD_v)/(3 - D_v)(V_{\text{max}}^{(3-D_v)/3} - V^{(3-D_v)/3}), \quad (5)$$

where V_s is the saturated water volume within a REV, V is the volume of the smallest pore filled, and V_{max} is the volume of the largest. Dividing by the volume of the REV and setting $V_0 = (FD_v) V_{\text{max}}^{(3-D_v)/3}/(3 - D_v)$ we get

$$\theta_s - \theta = V_0/V_{\text{REV}}[1 - (V/V_{\text{max}})^{(3-D_v)/3}], \quad (6)$$

which is essentially the relation derived by Perrier *et al.* [8]. Substituting from Eq. (4) we then get the relationship between media saturation and capillary pressure:

$$\theta_s - \theta = V_0/V_{\text{REV}}[1 - (P_s/P)^\lambda], \quad (7)$$

where $\lambda = (3 - D_v)/(3 - D_s)$, and P and P_s are the saturated and unsaturated capillary pressures. This is applicable to a

two phase system with wetting and nonwetting fluids. As with previous power-law formulations relating capillary to moisture content, Eq. (7) is likely to breakdown near saturation.

The results of Rieu and Sposito [7] and Tyler and Wheatcraft [6] can be obtained as special cases of Eq. (7). When the size of the smallest pore filled within the REV, V_{min}, is equal to 0, $V_0/V_{REV} = \theta_s$ and Eq. (7) also gives the well-known relationship presented by Brooks and Corey [1], where $\theta/\theta_s = (P_s/P)^\lambda$. This equation is sometimes written as $(\theta_s\theta)/(\theta_s - \theta_r) = (P_s/P)^\lambda$, where the left-hand side is termed the "reduced water content" with θ_r being the "residual water content" (i.e., the value below which vapor transport is required to reduce the water content further). The residual water content effectively represents the percolation threshold for the medium. The choice of plotting P_s/P against reduced water content stems less from the physics of the phenomena than because of a historical convention that was first used by Burdine ([25] and [26], p. 479) to force

relationships for hydraulic conductivity to zero at the percolation threshold.

IV. CONCLUSIONS

Many studies have measured the pore volume fractal dimension directly and the values obtained to date indicate that $3 - D_v < 1$ [27]. However, values of λ that are larger than this, and indeed greater than 2, have long been known to occur, indicating $\lambda = 3 - D_v$ is insufficient [1,27,28]. By considering a porous system with distinct pore surface and pore volume fractal dimensions we have derived a more general form of $\lambda = (3-D)/(3-D_s)$. When $D_s = 2$, the analysis reduces to the standard relationship, as expected, and when $D_s > 2$ it allows for a larger value of the exponent than $3 - D_v$. The results show that capillary pressure data alone are insufficient to establish the fractal nature of a pore space and that data on the pore surface fractal dimension are needed as well.

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